ON EXTREME POINTS IN l_1

BY

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ABSTRACT

It is proved that every bounded closed and convex subset of l_1 is the closed convex hull of its extreme points.

The well known Krein Milman theorem states that in a locally convex Hausdorff linear topological space X every compact convex set K is the closed convex hull of its extreme points. There are many known examples of non compact convex sets which are the closed convex hulls of their extreme points and hence in the formulation given above, the Krein Milman theorem does not characterize compact convex sets. If A is a compact set in a locally convex Hausdorff space X then the Krein Milman theorem implies that every closed convex *subset* K of A is the closed convex hull of its extreme points. It has been asked by several mathematicians how far does this property characterize compact sets A. In particular Ky Fan has asked in a symposium on linear spaces held in Jerusalem in 1964 whether in every non-quasireflexive Banach space there is a bounded closed convex set which does not have an extreme point. The recent important results of James [2] concerning the characterization of w compact sets may suggest that the answer to the question above is positive. However it turns out that the answer is negative. We prove here

THEOREM 1. *In the space l_1 every closed bounded and convex set is the closed convex hull of its extreme points.*

The question whether Theorem 1 in its present formulation is true was raised by M. A. Rieffel and was communicated to the author by R. R. Phelps.

In [3] we showed that l_1 has a closed subspace which is not isomorphic to a conjugate Banach space. Hence Theorem 1 implies

COROLLARY 1. *There is a Banach space X which is not isomorphic to a conjugate space such that every closed bounded and convex subset of X is the closed convex hull of its extreme points.*

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This corollary shows that the existence of many extreme points in every closed and convex subset of a given convex set A does not imply even a weak form of compactness of A.

The proof of Theorem 1 is based on the results of Bishop and Phelps [1].

LEMMA 1. *Let X be a Banach space. Then the following two statements are equivalent.*

(i) *Every bounded closed and convex subset of X has an extreme point.*

(ii) *Every bounded closed and convex subset of X is the closed convex hull of its extreme points.*

Proof. Clearly (ii) $\Rightarrow$ (i). Assume that (i) holds and let K be a bounded closed and convex subset of X. Let K_0 be the closed convex hull of the extreme points of K. If $K_0 \neq K$ then there is by [1] an $f \in X^*$ and a $y \in K$ such that

$$f(y) = \sup_{x \in K} f(x) > \sup_{x \in K_0} f(x) .$$

The set $K_1 = \{x; x \in K, f(x) = f(y)\}$ is a closed face of K which is disjoint from K_0. Since $K_0 \supset \operatorname{ext} K \supset \operatorname{ext} K_1 \neq \varnothing$ we get a contradiction. (ext K denotes the set of extreme points of K).

LEMMA 2. *Let K be a bounded closed and convex subset of l_1 and let $\varepsilon > 0$. Then there is a closed face F of K and an integer n such that $x = (x_1, x_2, \cdots) \in F$ $\Rightarrow \sum_{i>n} |x_i| \leqq \varepsilon$.*

Proof. Let $M = \sup_{x \in K} \| x \|$ and choose a $y = (y_1, y_2, \cdots)$ in K such that $\| y \| \geqq M - \varepsilon/4$. Let n be such that $\sum_{i=1}^{n} | y_i | \geqq M - \varepsilon/2$. and let $f \in l_1^*$ be defined by

$$f(x_1, x_2, \cdots) = \sum_{i=1}^{n} \operatorname{sgn}(y_i) x_i$$

(here $\operatorname{sgn} t = 1$ if $t \geqq 0$ and $= -1$ if $t < 0$). By [1] there is a $g \in l_1^*$ such that $\| f - g \| < \varepsilon/4M$ and $F = \{x; x \in K, g(x) = \sup_{u \in K} g(u)\}$ is not empty. We claim that F has the required properties. Indeed, let $x \in F$, then

$$\begin{aligned}\sum_{i=1}^{n} | x_i | \geqq f(x) &\geqq g(x) - \| f - g \| \, \| x \| \geqq g(y) - \| f - g \| \, \| x \| \\ &\geqq f(y) - \| f - g \| (\| x \| + \| y \|) \geqq M - \varepsilon/2 - \varepsilon/2.\end{aligned}$$

Hence, since $\| x \| \leqq M, \sum_{i>n} | x_i | \leqq \varepsilon$.

We are now ready to prove Theorem 1. Let K be a closed bounded and convex subset of l_1. By Lemma 1 we have only to show that $\operatorname{ext} K \neq \varnothing$. By Lemma 2 there is a sequence $\{F_i\}_{i=1}^{\infty}$ of closed faces of K and a sequence of integers $\{n_i\}_{i=1}^{\infty}$ such that F_{i+1} is a face of F_i and $x = (x_1, x_2, \cdots) \in F_i$ implies that $\sum_{j>n_i} | x_j | \leqq 1/i$. Let $\{y^i\}$ be a sequence of points in l_1 such that $y^i \in F_i$. From the properties of the F_i it follows immediately that the set $\{y^i\}_{i=1}^{\infty}$ is totally bounded. Hence since

K is closed $F = \bigcap_{i=1}^{\infty} F_i$ is a non empty compact face of K. By the Krein Milman theorem ext $F \neq \emptyset$ and hence ext $K \neq \emptyset$. This concludes the proof.

REMARK. The proof presented here can be used to prove the following more general result. Let $\{X_\alpha\}_{\alpha \in A}$ be a set of reflexive Banach spaces. Then every closed convex subset of $(\sum \oplus X_\alpha)_1$ is the closed convex hull of its extreme points. In this case the set F appearing in the proof of the theorem will be w compact (and not norm compact in general) but this suffices for the proof.

REFERENCES

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